THE SCHOOL FOR EXCELLENCE

UNIT 4 MATHEMATICAL METHODS 2006

COMPLIMENTARY WRITTEN EXAMINATION 1 - SOLUTIONS

QUESTION 1

a. (i) Divide $3x+1$ into $3x^4+4x^3-14x^2-20x-5$ using either synthetic or polynomial long division and get a remainder of zero:

$$\begin{array}{r|l} & x^3+x^2-5x-5 \\ \hline 3x+1 & 3x^4+4x^3-14x^2-20x-5 \\ & \underline{3x^4+x^3} \\ & \quad 3x^3-14x^2-20x-5 \\ & \quad \underline{3x^3+x^2} \\ & \qquad -15x^2-20x-5 \\ & \qquad \underline{-15x^2-5x} \\ & \qquad\qquad -15x-5 \\ & \qquad\qquad \underline{-15x-5} \\ & \qquad\qquad\qquad 0 \end{array}$$

(ii) It follows from (i) that:

$$p(x)=3x^4+4x^3-14x^2-20x-5=(3x+1)(x^3+x^2-5x-5).$$

b. (i) Using trial and error it is readily found that $q(-1)=0$.
It follows that $x-(-1)=x+1$ is a factor of $q(x)$.

(ii) Divide $x+1$ into $q(x)=x^3+x^2-5x-5$ using either synthetic or polynomial long division to get:

$$q(x)=x^3+x^2-5x-5=(x+1)(x^2-5)$$

Use difference of two squares formula:

$$=(x+1)(x-\sqrt{5})(x+\sqrt{5}).$$

Therefore: $p(x)=(3x+1)q(x)=(3x+1)(x+1)(x-\sqrt{5})(x+\sqrt{5})$.

QUESTION 2

a. By definition: $\int_0^b ax\,dx = 1$.

Therefore: $\frac{a}{2}\left[x^2\right]_0^b = 1$

$\therefore \frac{a}{2}b^2 = 1 \Rightarrow ab^2 = 2$.

b. By definition: $\overline{X} = \int_0^b (x)ax\,dx = a\int_0^b x^2\,dx$.

Therefore: $\overline{X} = \frac{a}{3}\left[x^3\right]_0^b = \frac{a}{3}b^3$.

c. Using the result from part (b): $\frac{a}{3}b^3 = 1 \therefore ab^3 = 3$. (1)

From part (a): $ab^2 = 2$. (2)

Equation (1)/Equation (2): $b = \frac{3}{2}$.

Substitute $b = \frac{3}{2}$ into equation (2): $a\left(\frac{3}{2}\right)^2 = 2 \Rightarrow a = \frac{8}{9}$.

QUESTION 3

Substitute $w = 2^x$. Then: $2^{3x} + 2^{2x} - 2^{-x} = 1 \Leftrightarrow (2^x)^3 + (2^x)^2 - (2^x)^{-1} = 1$

$\therefore w^3 + w^2 - w^{-1} = 1$

Multiply both sides by *w*: $\therefore w^4 + w^3 - 1 = w \Leftrightarrow w^4 + w^3 - w - 1 = 0$

$(w^4 - 1) + (w^3 - w) = 0$

$(w^2 - 1)(w^2 + 1) + w(w^2 - 1) = 0$

$(w^2 - 1)(w^2 + w + 1) = 0$.

Use the null factor theorem: $w^2 + w + 1 = 0$ or $w^2 - 1 = 0$.

$w^2 + w + 1 = 0$ has no real solution.

$w^2 - 1 = 0 \Rightarrow w = \pm 1$.

Back-substitute $w = 2^x$: $2^x = \pm 1$.

$2^x = -1$ has no real solution.

$2^x = 1 \Rightarrow 2^x = 2^0 \Rightarrow x = 0$.

QUESTION 4

Let $y = f^{-1}(t)$. Then: $t = \dfrac{3y}{y^2+1} \therefore t(y^2+1) = 3y \therefore ty^2 - 3y + t = 0$.

Use the quadratic formula to solve for y: $\quad y = \dfrac{3 \pm \sqrt{9-4t^2}}{2t}$.

To choose between the two potential solutions for y, use the fact that

$f(a) = b \Rightarrow f^{-1}(b) = a$.

For example, since $f(0) = 0$ it is required that $f^{-1}(0) = 0$.

Since $\dfrac{3+\sqrt{9-4(0)^2}}{2(0)}$ is undefined, it follows by elimination that $y = f^{-1}(t) = \dfrac{3-\sqrt{9-4t^2}}{2t}$.

Note: $\lim\limits_{t\to 0} \dfrac{3-\sqrt{9-4t^2}}{2t} = 0$.

QUESTION 5

a. Let $y = \dfrac{u}{v}$ where $u = \cos^2 x$ and $v = x$.

Then $\dfrac{dv}{dx} = 1$.

To find $\dfrac{du}{dx}$ use the **Chain Rule:**

Let $w = \cos x$ so that $u = w^2$.

Then $\dfrac{du}{dw} = 2w$ and $\dfrac{dw}{dx} = -\sin x$.

Then $\dfrac{du}{dx} = \dfrac{du}{dw} \times \dfrac{dw}{dx} = (2w)(-\sin x) = -2\cos x \sin x$.

From the **Quotient Rule:**

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{(x)(-2\cos x \sin x) - (\cos^2 x)(1)}{x^2} = \frac{-2x\sin x\cos x - \cos^2 x}{x^2}.$$

b. To find x-coordinate of stationary points solve $\frac{dy}{dx} = 0$:

$$\frac{dy}{dx} = \frac{-2x\sin x\cos x - \cos^2 x}{x^2} = \frac{-\cos x(2x\sin x + \cos x)}{x^2} = 0$$

Use the null factor theorem:

$\therefore \cos x = 0$ or $2x\sin x + \cos x = 0$.

Solve for x: $\cos x = 0$

$$x = \frac{(2n+1)\pi}{2} \text{ where } n \in J \text{ .}$$

Solve for y: $\cos x = 0$

$$y = \frac{2(0)^2}{(2n+1)\pi} = 0$$

c. $2x\sin x + \cos x = 0$

$\therefore 2x\sin x = -\cos x$

$\therefore 2x\frac{\sin x}{\cos x} = -1$

$\therefore 2x\tan x = -1$.

QUESTION 6

Let the point where $y = \frac{x}{2} - 1$ is normal to $y = bx^4 + 1$ have coordinates $(x_1,\ y_1)$.

Gradient of normal to curve at $(x_1,\ y_1)$:

$$m_{\text{tangent}} = \frac{dy}{dx} = 4bx^3.$$

Therefore $m_{\text{tangent}} = 4bx_1^{\ 3}$ at $(x_1,\ y_1)$.

$$m_{\text{normal}}\, m_{\text{tangent}} = -1 \therefore m_{\text{normal}} = -\frac{1}{m_{\text{tangent}}} \therefore m_{normal} = -\frac{1}{4bx_1^{\ 3}}. \qquad (1)$$

Gradient of line: $m = \frac{1}{2}$. (2)

Equate equations (1) and (2): $-\frac{1}{4bx_1^{\ 3}} = \frac{1}{2} \Rightarrow 2bx_1^{\ 3} = -1$

$$\therefore 2bx_1^{\ 3} = -1. \qquad (3)$$

Since the point $(x_1,\ y_1)$ lies on both the line and the curve, it follows that

$$y_1 = \frac{x_1}{2} - 1. \qquad (4)$$

$$y_1 = bx_1^{\ 4} + 1. \qquad (5)$$

Equate equations (4) and (5):

$$\frac{x_1}{2} - 1 = bx_1^{\ 4} + 1 \therefore 2bx_1^{\ 4} - x_1 + 4 = 0. \qquad (6)$$

From equation (3) it follows that $2bx_1^{\ 4} = -x_1$.

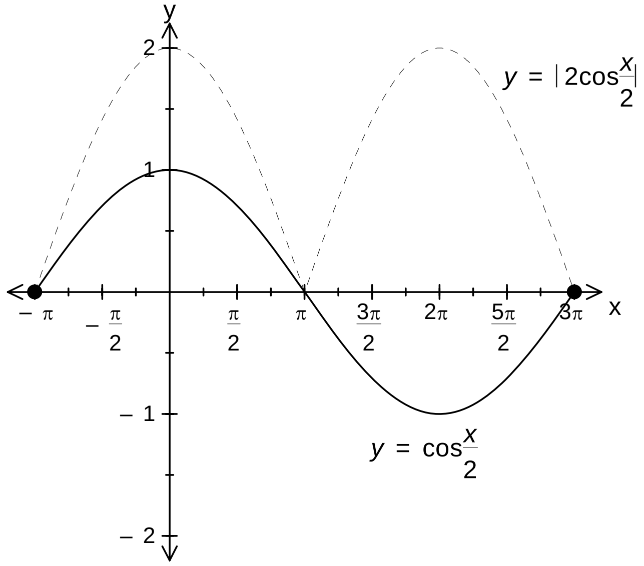
Substitute $2bx_1^{\ 4} = -x_1$ into equation (6): $-x_1 - x_1 + 4 = 0 \therefore x_1 = 2$.

Substitute $x_1 = 2$ into equation (4): $y_1 = 0$.

Substitute $x_1 = 2$ and $y_1 = 0$ into equation (5): $0 = b(2)^4 + 1 \quad \therefore b = -\frac{1}{16}$.

QUESTION 7

a. and **b.**



c. Note that $y = |2f(x)| = \left|2\cos\left(\frac{x}{2}\right)\right| = \begin{cases} 2\cos\left(\frac{x}{2}\right) & \text{for } -\pi \le x \le \pi \\ -2\cos\left(\frac{x}{2}\right) & \text{for } \pi \le x \le 3\pi \end{cases}$

Therefore:

$$\text{Area} = \int_{-\pi}^{\pi} 2\cos\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right) dx + \int_{\pi}^{3\pi} -2\cos\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right) dx$$

$$= \int_{-\pi}^{\pi} \cos\left(\frac{x}{2}\right) dx - 3\int_{\pi}^{3\pi} \cos\left(\frac{x}{2}\right) dx$$

$$= 2\left[\sin\left(\frac{x}{2}\right)\right]_{-\pi}^{\pi} - 6\left[\sin\left(\frac{x}{2}\right)\right]_{\pi}^{3\pi}$$

$$= 2\left\{\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right\} - 6\left\{\sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right\} = 2\{1-(-1)\} - 6\{-1-1\} = 16.$$

QUESTION 8

$g(x) = f(-2x) + 1$.

Transformations: Reflection in the vertical axis.

Dilation by a scale factor of $\frac{1}{2}$ from the vertical axis.

Translation by +1 units from the *x*-axis.

The order in which these transformations are applied does not matter.

QUESTION 9

a. Volume of a right circular cone: $V = \frac{1}{3}\pi r^2 h$.

Semi-vertical angle of $45^0 \Rightarrow \tan 45^0 = \frac{r}{h} \Rightarrow r = h$.

Therefore: $V = \frac{1}{3}\pi h^3$.

After 30 minutes $V = (30)(0 \cdot 3) = 9$ m^3.

Therefore: $9 = \frac{1}{3}\pi h^3$

$h^3 = \frac{27}{\pi}$

$h = \frac{3}{\pi^{1/3}}$ meters.

b. From the chain rule: $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$.

Given: $\frac{dV}{dt} = 0 \cdot 3$ m^3/min.

From part (a): $V = \frac{1}{3}\pi h^3 \therefore \frac{dV}{dh} = \pi h^2 \therefore \frac{dh}{dV} = \frac{1}{\pi h^2}$.

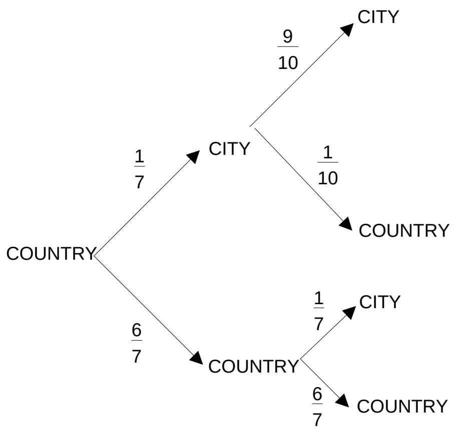
Therefore: $\frac{dh}{dt} = \frac{1}{\pi h^2} \times 0 \cdot 3$.

Substitute $t = 30$ minutes $\Rightarrow h = \frac{3}{\pi^{1/3}}$ meters:

$$\frac{dh}{dt} = \frac{1}{\pi\left(\frac{3}{\pi^{1/3}}\right)^2} \times 0 \cdot 3 = \frac{0 \cdot 3\pi^{1/3}}{9\pi} = \frac{1}{30\pi^{1/3}} \text{ m/min.}$$

QUESTION 10

Draw a tree diagram:



Pr(Living in the city in 2 years time | currently living the in country)

$$= \left(\frac{1}{7}\right)\left(\frac{9}{10}\right) + \left(\frac{6}{7}\right)\left(\frac{1}{7}\right)$$

$$= \frac{9}{70} + \frac{6}{49} = \frac{63}{490} + \frac{60}{490} = \frac{123}{490}.$$

QUESTION 11

a. By definition: $p + \frac{1}{7} + 3p + 3p + 4p + p = 1$

$$\therefore 12p = \frac{6}{7}$$

$$\therefore p = \frac{1}{14}.$$

b. Conditional probability: Require $\Pr(X < 5 \mid X > 2)$.

x	1	2	3	4	5	6
$\Pr(X = x)$	p	$\frac{1}{7}$	$3p$	$3p$	$4p$	p

$$\Pr(X < 4 \mid X > 2) = \frac{\Pr(X = 3) + \Pr(X = 4)}{\Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5) + \Pr(X = 6)} = \frac{6p}{11p} = \frac{6}{11}.$$

BONUS QUESTION

QUESTION 12

a. Let $y = uv$ where $u = x$ and $v = e^{-x}$.

Then $\dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = -e^{-x}$.

From the Product Rule: $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$\therefore \dfrac{dy}{dx} = (x)(-e^{-x}) + (e^{-x})(1) = e^{-x} - xe^{-x}$.

b. Anti-differentiate both sides of $\dfrac{dy}{dx} = e^{-x} - xe^{-x}$ with respect to *x*:

$y = \int (e^{-x} - xe^{-x})\, dx$

Substitute $y = xe^{-x}$:

$\therefore xe^{-x} = \int e^{-x}\, dx - \int xe^{-x}\, dx = -e^{-x} - \int xe^{-x}\, dx$

$\therefore \int xe^{-x}\, dx = -xe^{-x} - e^{-x} = -(x+1)e^{-x}$,

Note: The arbitrary constant of anti-differentiation is omitted since only *an* anti-derivative is required.

c. (i) By definition: $\overline{X} = \int_0^{\log_e 3} (x)\dfrac{3}{2}e^{-x}\, dx = \dfrac{3}{2}\int_0^{\log_e 3} xe^{-x}\, dx$.

Substitute the result from part (b):

$\overline{X} = -\dfrac{3}{2}\Big[(x+1)e^{-x}\Big]_0^{\log_e 3} = -\dfrac{3}{2}\left\{(1+\log_e 3)e^{-\log_e 3} - 1\right\}$

Use the log rule $\log_a \dfrac{1}{b} = -\log_a b$:

$= \dfrac{3}{2}\left\{1 - (1+\log_e 3)e^{\log_e \frac{1}{3}}\right\}$

Use the log rule $a^{\log_a b} = b$:

$$= \frac{3}{2}\left\{1 - \frac{1}{3}(1 + \log_e 3)\right\} = 1 - \frac{1}{2}\log_e 3$$

Use the log rule $c\log_a b = \log_a b^c$ and the power rule $b^{1/2} = \sqrt{b}$:

$$= 1 - \log_e \sqrt{3}\ .$$

(ii) Let the median value of X be equal to k .

By definition: $\frac{1}{2} = \int_0^k \frac{3}{2} e^{-x}\, dx$.

Therefore:

$$\frac{1}{3} = \int_0^k e^{-x}\, dx = -\left[e^{-x}\right]_0^k = 1 - e^{-k}$$

$$\therefore \frac{2}{3} = e^{-k}$$

$$-k = \log_e \frac{2}{3} \qquad .$$

$$k = -\log_e \frac{2}{3} = \log_e \frac{3}{2}$$