Insight Publications

INSIGHT

Trial Exam Paper

2007

SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions.
- mark allocations
- tips on how to approach the questions.

Question 1

Let $x = \sqrt{t+4}$ and $y = 1 - t$ for $-4 \le t \le 4$.

1a. Find the Cartesian equation of the curve.

Worked solution

$x = \sqrt{t+4}$ ….. (1) for $-4 \le t \le 4$
$y = 1 - t$ ….. (2)

From (1) $x^2 = t + 4$

$t = x^2 - 4$ 1A

Substitute into (2)

$y = 1 - \left(x^2 - 4\right)$

$y = 5 - x^2$ 1A

2 marks

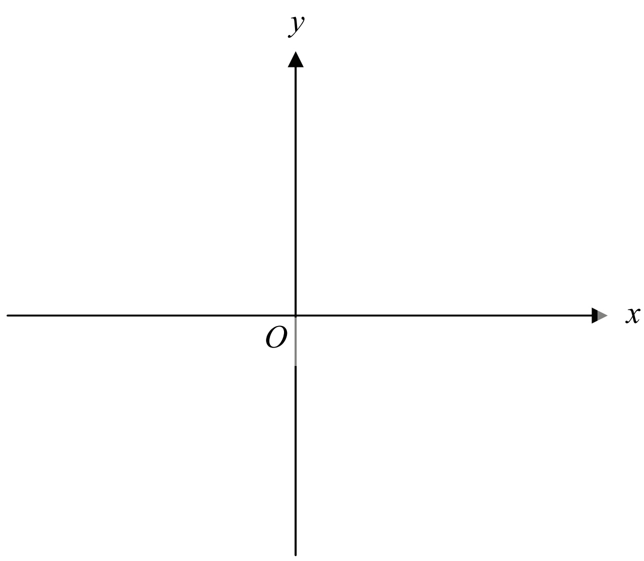
Mark allocation

- 1 mark for correctly expressing t in terms of x^2
- 1 mark for the correct answer

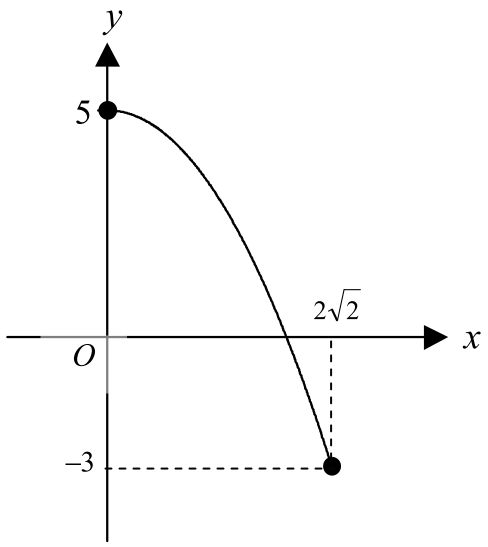
Tip

- *Use substitution to eliminate* t *from the parametric equations.*

1b. Sketch a graph of the curve showing all features clearly.



Worked solution



2 marks

Mark allocation

- 1 mark for correct shape
- 1 mark for both endpoints correct

Tip

- *Use the restrictions on t to find the endpoints*

  When $t = -4$ $\quad x = \sqrt{-4+4} = 0$

  $y = 1 - (-4) = 5$

  When $t = 4$ $\quad x = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

  $y = 1 - 4 = -3$

Question 2

Express $\left(\sqrt{3}-i\right)^7$ in the form $x+iy$ where $x,\ y \in R$

Worked solution

Let $r\text{cis}\theta = \left(\sqrt{3}-i\right)$	Express $\sqrt{3}-i$ in polar form	
$r=\sqrt{\left(\sqrt{3}\right)^2+1^2}=\sqrt{4}=2$		
$\tan\theta = \dfrac{-1}{\sqrt{3}}$		
$\theta = \dfrac{11\pi}{6}$	Fourth quadrant angle	
$\therefore \theta = -\dfrac{\pi}{6}$	Calculate equivalent angle $\theta \in (-\pi, \pi]$	
$\left(\sqrt{3}-i\right)=2\text{cis}\left(-\dfrac{\pi}{6}\right)$		1A
$\left(\sqrt{3}-i\right)^7 = 2^7\text{cis}\left(7\times -\dfrac{\pi}{6}\right)$	Applying De Moivre's theorem	1A
$=128\text{cis}\left(-\dfrac{7\pi}{6}\right)$		
$=128\left(\cos\left(-\dfrac{7\pi}{6}\right)+i\sin\left(-\dfrac{7\pi}{6}\right)\right)$		
$=128\left(\cos\left(\dfrac{7\pi}{6}\right)-i\sin\left(\dfrac{7\pi}{6}\right)\right)$		
$=128\left(-\cos\left(\dfrac{\pi}{6}\right)-i\times -\sin\left(\dfrac{\pi}{6}\right)\right)$		1M
$=128\left(-\cos\left(\dfrac{\pi}{6}\right)+i\sin\left(\dfrac{\pi}{6}\right)\right)$		
$=128\left(-\dfrac{\sqrt{3}}{2}+i\times\dfrac{1}{2}\right)$		
$=-64\sqrt{3}+64i$		1A

4 marks

Mark allocation

- 1 mark for expressing $\sqrt{3}-i$ in correct polar form
- 1 mark for correctly applying de Moivre's theorem
- 1 mark for correct method of simplification
- 1 mark for correct answer

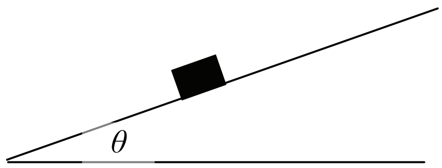
Tip

- *Express complex number in polar form and then apply De Moivre's theorem*

Question 3

A 10 kg mass is pulled up a rough plane inclined at an angle of θ to the horizontal by a force of 120 newtons acting parallel to the plane.

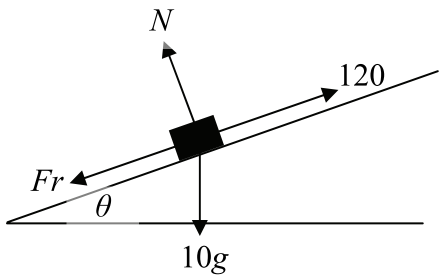
The coefficient of friction between the mass and the plane is $\frac{1}{3}$, $\cos(\theta) = \frac{3}{5}$ and the acceleration due to gravity is g m/s^2.



3a. Show all forces acting on the mass on the diagram above.

Worked solution

N = normal reaction
Fr = frictional force



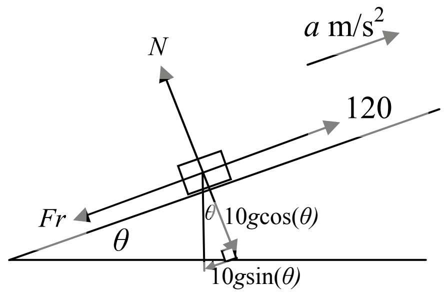
1 mark

Mark allocation

- 1 mark for all forces correctly shown

3b. Find the acceleration of the mass up the plane in terms of g.

Worked solution



Equation of motion up the plane

$R = ma$ — R = resultant force — $Fr = \mu N$

$120 - Fr - mg\sin(\theta) = ma$ — $\mu = \frac{1}{3}$ = coefficient of friction

$120 - \mu N - 10g\sin(\theta) = 10a$ ….(1) — 1A

Resolving forces perpendicular to the plane

$N = 10g\cos(\theta)$ ….(2)

Substitute (2) into (1)

$120 - \frac{1}{3}\times 10g\cos(\theta) - 10g\sin(\theta) = 10a$ — 1M

$a = 12 - \frac{1}{3}\times g\cos(\theta) - g\sin(\theta)$

Given $\cos(\theta) = \frac{3}{5}$

$\sin(\theta) = \sqrt{1-\cos^2(\theta)} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$ — 1A

$a = 12 - \frac{1}{3}\times g\times\frac{3}{5} - g\times\frac{4}{5}$

$a = 12 - \frac{1}{5}g - \frac{4}{5}g$

$a = 12 - \frac{5}{5}g$

$a = 12 - g$ m/s^2 — 1A

4 marks

Mark allocation

- 1 mark for correctly resolving forces parallel to the plane
- 1 mark for using a correct method to give the acceleration in terms of θ and g.
- 1 mark for correctly finding $\sin\theta = \frac{4}{5}$
- 1 mark for correct answer

Tips

- *Resolve forces parallel and perpendicular to the plane*
- $\sin\theta$ *is not given and should be found from value of* $\cos\theta$

Question 4

4a. Show that $\dfrac{\sin(x)}{1-\cos(x)} = \cot\left(\dfrac{x}{2}\right)$

Worked solution

$$\text{LHS} = \frac{\sin(x)}{1-\cos(x)}$$

$$= \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{1-\cos\left(2\left(\frac{x}{2}\right)\right)} \qquad \text{1M}$$

$$= \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{1-\left(1-2\sin^2\left(\frac{x}{2}\right)\right)}$$

$$= \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)}$$

$$= \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \qquad \text{1A}$$

$$= \cot\left(\frac{x}{2}\right)$$

$$= \text{RHS}$$

2 marks

Mark allocation

- 1 mark for correct application of the double angle formulae
- 1 mark for correct simplification leading to the correct answer

Tips

- *Express* $\sin(x)$ *and* $\cos(x)$ *in terms of* $\frac{\theta}{2}$ *using double angle formulas.*

4b. **Hence** or otherwise solve the equation $\sin(x) = \cos(x) - 1$ over $0 \le x \le 2\pi$

Worked solution

$$\frac{\sin(x)}{\cos(x)-1} = 1$$

$$\frac{\sin(x)}{1-\cos(x)} = -1 \qquad \text{1A}$$

$$\cot\left(\frac{x}{2}\right) = -1$$

$$\tan\left(\frac{x}{2}\right) = -1$$

$$\frac{x}{2} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$x = \frac{3\pi}{2} \qquad \text{1A}$$

2 marks

Mark Allocation

- 1 mark for rearranging the equation to use information given in part a of the question
- 1 mark for the correct answer

Tip

- *The word 'hence' gives the hint that something from the previous part of the question is used to find the answer.*

Question 5

The position of a particle at time t seconds, $t \geq 0$, is given by the vector $\underset{\sim}{r} = t\underset{\sim}{i} + (1-2t)\underset{\sim}{j} + (t-6)\underset{\sim}{k}$. Find the time when the particle's velocity vector is perpendicular to its position vector.

Worked solution

Position vector: $\underset{\sim}{r} = t\underset{\sim}{i} + (1-2t)\underset{\sim}{j} + (t-6)\underset{\sim}{k}$

Differentiate to find velocity vector: $\dot{\underset{\sim}{r}} = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ 1A

Vectors are perpendicular when $\underset{\sim}{r}.\dot{\underset{\sim}{r}} = 0$

$\underset{\sim}{r}.\dot{\underset{\sim}{r}} = \left(t\underset{\sim}{i} + (1-2t)\underset{\sim}{j} + (t-6)\underset{\sim}{k}\right).\left(\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right) = 0$ 1A

$\underset{\sim}{r}.\dot{\underset{\sim}{r}} = t - 2(1-2t) + (t-6) = 0$

$t - 2 + 4t + t - 6 = 0$

$6t - 8 = 0$

$t = \frac{4}{3}$ seconds 1A

3 marks

Mark allocation

- 1 mark for finding the correct velocity vector
- 1 mark for taking the dot product of the position and velocity vectors and setting this to zero.
- 1 mark for the correct answer

Tip

- *The dot product is zero when two vectors are perpendicular.*

Question 6

Consider the relation $xy + \frac{y^2}{x} = 2$.

6a. Find an expression for $\frac{dy}{dx}$ in terms of x and y.

Worked solution

$xy + \frac{y^2}{x} = 2$

$x^2 y + y^2 = 2x$

$2xy + x^2 \frac{dy}{dx} + 2y \frac{dy}{dx} = 2$ 1M

$\frac{dy}{dx}\left(x^2 + 2y\right) = 2 - 2xy$ 1A

$\frac{dy}{dx} = \frac{2(1 - xy)}{\left(x^2 + 2y\right)}$ 1A

3 marks

Mark allocation

- 1 mark for correctly applying implicit differentiation techniques
- 1 mark for factorising correctly to collect the terms containing $\frac{dy}{dx}$
- 1 mark for correct answer

Tip

- *The quotient rule can be used to find the answer, however, it is much easier to first simplify as shown and use the product rule.*

Question 6 – continued

6b. Hence find the equations of the tangents to the curve when $x = 1$

Worked solution

Finding y when $x = 1$

$$xy + \frac{y^2}{x} = 2$$

$$1y + \frac{y^2}{1} = 2$$

$$y^2 + y - 2 = 0$$

$$(y+2)(y-1) = 0$$

$y = -2$, $y = 1$ 1A

Gradient of tangent at $x = 1$, $y = -2$

$$\frac{dy}{dx} = \frac{2(1-xy)}{(x^2+2y)} = \frac{2(1-1\times -2)}{1^2+2\times -2} = \frac{6}{-3} = -2$$

Equation of tangent at $x = 1$, $y = -2$

$$y-(-2) = -2(x-1)$$

$y = -2x$ 1A

Gradient of tangent at $x = 1$, $y = 1$

$$\frac{dy}{dx} = \frac{2(1-xy)}{(x^2+2y)} = \frac{2(1-1\times 1)}{1^2+2\times -1} = \frac{0}{-1} = 0$$

Equation of tangent at $x = 1$, $y = 1$

$y = 1$ 1A

3 marks

Mark Allocation

- 1 mark for finding the y-coordinates of the points where $x = 1$
- 1 mark for finding the correct equation of one tangent
- 1 mark for finding the other tangent equation

Tips

- *Recognise there will be two tangents from the wording of the question.*
- *A line with zero gradient will be parallel to the x-axis*

Question 7

$f : D \to R, \quad f(x) = \arccos\left(\frac{1}{\sqrt{x}}\right)$

7a. Determine the domain D of function f

Worked solution

$-1 \le \frac{1}{\sqrt{x}} \le 1$

$\Rightarrow \quad -1 \le \frac{1}{\sqrt{x}} \quad$ and $\quad \frac{1}{\sqrt{x}} \le 1$

$-\sqrt{x} \le 1 \quad$ and $\quad 1 \le \sqrt{x}$

$\sqrt{x} \ge -1 \quad$ and $\quad x \ge 1$

$x \ge (-1)^2$

$\therefore \quad x \ge 1$

Domain of f, $D = [1, \infty)$ 1A

1 mark

Mark allocation

- 1 mark for correct domain.

7b. Find $f'(x)$

Worked solution

Let $y = \arccos\left(\frac{1}{\sqrt{x}}\right) \qquad u = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$y = \arccos(u) \qquad \frac{du}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$

$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}} \times \left(-\frac{1}{2}x^{-\frac{3}{2}}\right)$ 1A

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1}{\sqrt{x}}\right)^2}} \times \frac{-1}{2\sqrt{x^3}}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\frac{1}{x}}} \times \frac{-1}{2x\sqrt{x}}$ 1M

$\therefore f'(x) = \frac{dy}{dx} = \frac{1}{2x\sqrt{x-1}}$ 1A

3 marks

Question 7 – continued

Mark allocation

- 1 mark for correctly applying the chain rule
- 1 mark for some correct simplification
- 1 mark for answer fully simplified

Tip

- *The chain rule needs to be used.*

Question 8

Solve the differential equation $\frac{dt}{dx} = \frac{t^2+3}{t^2}$ given $x = 1$ when $t = 1$

Worked solution

$$\frac{dx}{dt} = \frac{t^2}{t^2+3}$$

$$\frac{dx}{dt} = 1 - \frac{3}{t^2+3} \qquad \text{1A}$$

$$x = \int 1 - \frac{3}{t^2+3}\,dt$$

$$x = t - \int \frac{3}{t^2+3}\,dt$$

$$x = t - \sqrt{3}\int \frac{\sqrt{3}}{t^2+3}\,dt$$

$$x = t - \sqrt{3}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c \qquad \text{1A}$$

When $t = 1$, $x = 1$

$$\Rightarrow \quad 1 = 1 - \sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + c$$

$$\Rightarrow \quad c = \sqrt{3} \times \frac{\pi}{6}$$

$$\therefore \quad x = t - \sqrt{3}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + \frac{\sqrt{3}\pi}{6} \qquad \text{1A}$$

3 marks

Mark allocation

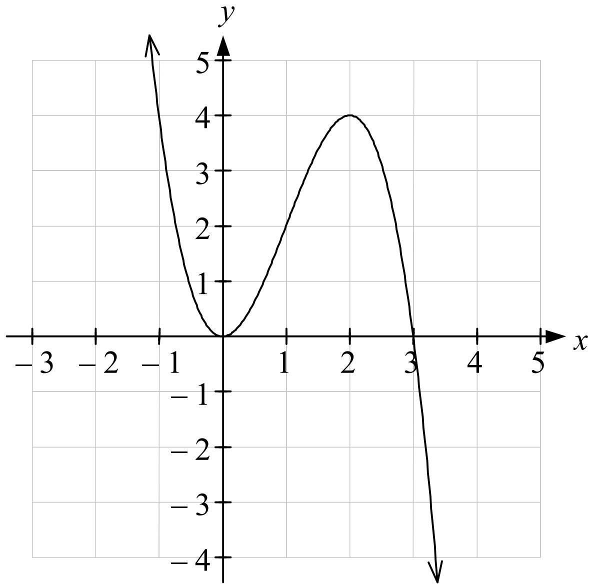
- 1 mark for converting the differential equation to $\frac{dx}{dt}$ and dividing t^2 by t^2+3.
- 1 mark for correct integration
- 1 mark for finding the correct answer

Tips

- *Recognise to take the reciprocal of the differential equation*
- *The numerator needs to be of lower degree than the denominator*

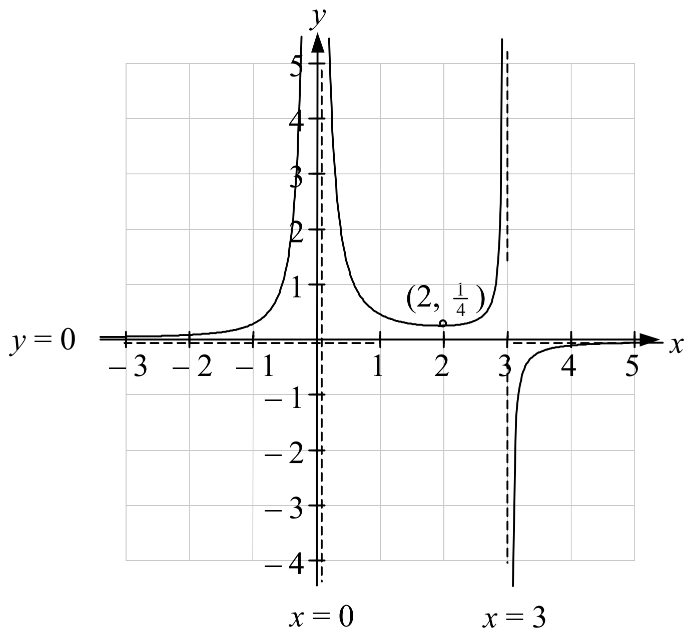
Question 9

The graph of $f(x) = 3x^2 - x^3$ is shown on the axes below.



9a. Draw the graph of $g(x) = \dfrac{1}{3x^2 - x^3}$ on the axes above showing all features clearly.

Worked solution



2 marks

Mark allocation

- 1 mark for correct shape
- 1 mark for correct asymptotes and turning point

Question 9 – continued

9b. Given $\frac{1}{3x^2 - x^3} = \frac{Ax + B}{x^2} + \frac{C}{3 - x}$. Find the exact values of A, B, and C.

Worked solution

$$\frac{1}{3x^2 - x^3} = \frac{1}{x^2(3-x)} = \frac{Ax+B}{x^2} + \frac{C}{3-x}$$

$$\frac{1}{x^2(3-x)} = \frac{Ax+B}{x^2} + \frac{C}{3-x}$$

$$\frac{1}{x^2(3-x)} = \frac{(Ax+B)(3-x)}{x^2(3-x)} + \frac{Cx^2}{x^2(3-x)}$$

$1 = 3Ax - Ax^2 + 3B - Bx + Cx^2$

$1 = (C - A)x^2 + (3A - B)x + 3B$ 1M

Equating coefficients of powers of x

$0 = C - A$

$0 = 3A - B$

$1 = 3B$

$\therefore \quad B = \frac{1}{3}, \quad A = \frac{1}{9}, \quad C = \frac{1}{9}$ 1A

2 marks

Mark allocation

- 1 mark for correct simplifications leading to simultaneous equations in A, B, C
- 1 mark for correct values of A, B, C

Question 9 – continued

9c. Find the exact area between the graph of $g(x) = \frac{1}{3x^2 - x^3}$, the x-axis and the lines $x = 1$ and $x = 2$

Worked solution

$$\text{Area} = \int_1^2 \frac{1}{3x^2 - x^3}\, dx$$

$$= \int_1^2 \frac{x+3}{9x^2} + \frac{1}{9(3-x)}\, dx \qquad 1A$$

$$= \frac{1}{9}\int_1^2 \frac{x}{x^2} + \frac{3}{x^2} + \frac{1}{(3-x)}\, dx$$

$$= \frac{1}{9}\left[\log_e|x| - 3x^{-1} - \log_e|3-x|\right]_1^2 \qquad 1M$$

$$= \frac{1}{9}\left[\log_e\left|\frac{x}{3-x}\right| - \frac{3}{x}\right]_1^2$$

$$= \frac{1}{9}\left[\left(\log_e(2) - \frac{3}{2}\right) - \left(\log_e\left(\frac{1}{2}\right) - 3\right)\right]$$

$$= \frac{1}{9}\left[\log_e(2) - \frac{3}{2} - \log_e\left(\frac{1}{2}\right) + 3\right]$$

$$= \frac{1}{9}\left[\log_e(2) + \log_e(2) + \frac{3}{2}\right]$$

$$= \frac{1}{9}\left[2\log_e(2) + \frac{3}{2}\right]$$

$$= \frac{2}{9}\log_e(2) + \frac{1}{6} \qquad 1A$$

3 marks

Mark allocation

- 1 mark for writing the correct integral and recognising the need to express as partial fractions
- 1 mark for using correct methods to integrate
- 1 mark for the correct exact answer

Tip

- *The quotient can be simplified in the following way to make the integration easier*

$$\frac{1}{3x^2 - x^3} = \frac{\frac{1}{9}x + \frac{1}{3}}{x^2} + \frac{\frac{1}{9}}{3-x}$$

$$\Rightarrow \quad \frac{1}{3x^2 - x^3} = \frac{x+3}{9x^2} + \frac{1}{9(3-x)}$$

END OF WORKED SOLUTIONS BOOK