The Mathematical Association of Victoria

SPECIALIST MATHEMATICS

Trial written examination 1

2007

Reading time: 15 minutes
Writing time: 1 hour

Student's Name: ....

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
8	8	40

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

These questions have been written and published to assist students in their preparations for the 2007 Specialist Mathematics Examination 1. The questions and associated answers and solutions do not necessarily reflect the views of the Victorian Curriculum and Assessment Authority. The Association gratefully acknowledges the permission of the Authority to reproduce the formula sheet.

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Working space

Instructions

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the acceleration due to gravity to have magnitude g m/s^2, where $g = 9.8$.

Question 1

Given that $P(z) = z^4 - 4z^3 + 6z^2 - 4z + 5 = 0$ and $P(2+i) = 0$, find all the roots of $P(z) = 0$.

3 marks

Question 2

$\underset{\sim}{u}$ and $\underset{\sim}{v}$ are vectors defined by $\underset{\sim}{u} = \cos(\theta)\underset{\sim}{i} + \sin(\theta)\underset{\sim}{j}$, $\underset{\sim}{v} = \sin(\theta)\underset{\sim}{i} + \cos(\theta)\underset{\sim}{j}$ and $0 < \theta < \frac{\pi}{2}$.

a. Show that $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are unit vectors.

1 mark

b. Let α be the angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$. Express α in terms of θ.

1 mark

c. Find α when $\theta = \frac{\pi}{6}$.

1 mark

d. If $\theta = \frac{\pi}{3}$, find the vector resolute of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{u}$.

2 marks

Question 3

a. Express $\dfrac{x+2}{x^2+x}$ in partial fractions with integer numerators.

2 marks

b. Hence show that $\displaystyle\int_{-4}^{-3} \frac{x+2}{x^2+x}dx = \log_e\left(\frac{a}{b}\right)$ where a and b are positive integers.

Find the values of a and b.

3 marks

Question 4

a. Show that, for $0 < x < \frac{1}{3}$, $\frac{d}{dx}\left(\cos^{-1}\left(\sqrt{(3x)}\right)\right) = \frac{-3}{2\sqrt{3x(1-3x)}}$

2 marks

b. **Hence**, find the exact value of $\int_{\frac{1}{12}}^{\frac{1}{6}} \frac{1}{\sqrt{3x(1-3x)}}dx$

3 marks

Question 5

An object of mass 2 kg falls from rest from a height of 50 metres. Its fall is opposed by an air resistance of magnitude of $0.05v^2$ newton, where v is its velocity.

a. Write an equation of motion for the falling object.

1 mark

b. Show that $\frac{dx}{dv} = \frac{40v}{40g - v^2}$

2 marks

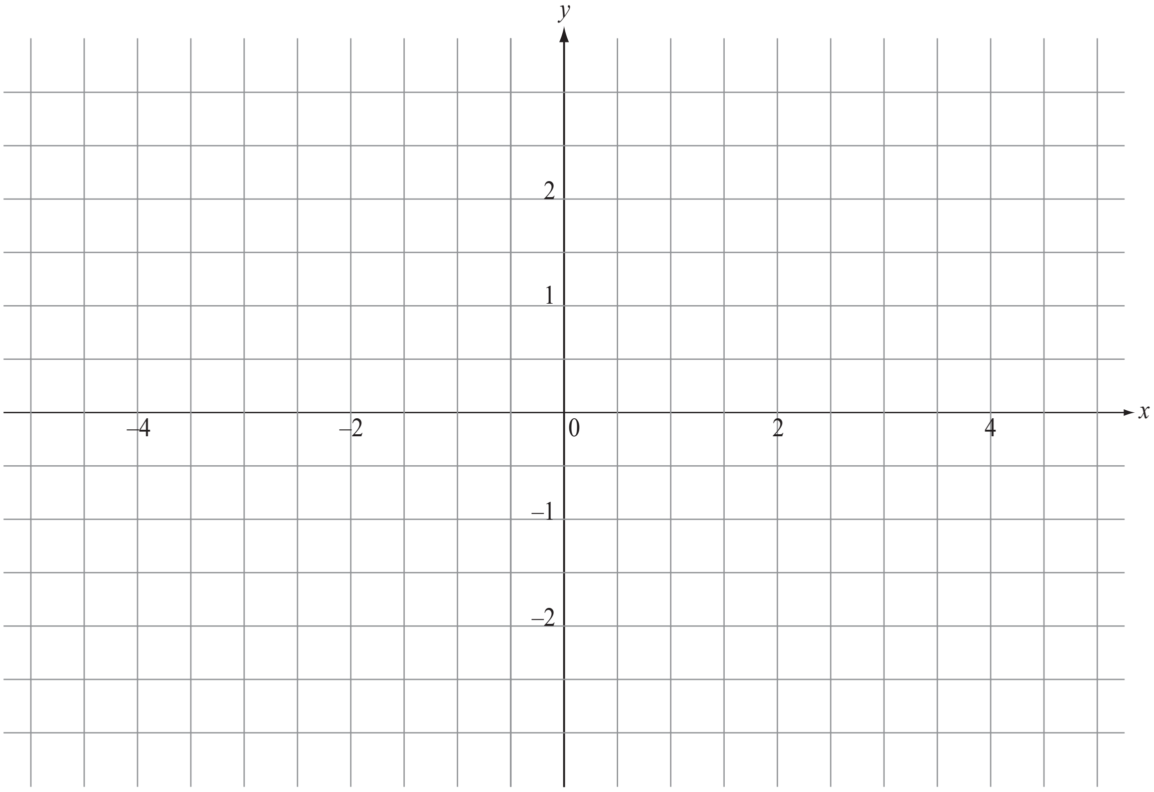
c. Hence, find the exact distance travelled for the object to reach a speed of 10 m/s.

3 marks

Question 6

Let $f(x) = \arctan(x) + \frac{\pi}{4}$, $x \in R$.

a. On the axes below, sketch the graph of $f(x)$. On the sketch, clearly indicate the asymptotes and axes intercepts.



3 marks

b. Solve $f(x) = \frac{5\pi}{12}$

1 mark

Question 7

At time t seconds, a particle has position vector

$$\underset{\sim}{r} = (3\cos(t) - \sin(2t))\underset{\sim}{i} + (3\sin(t) + \cos(2t))\underset{\sim}{j}, \text{ where } t \geq 0.$$

a. Find its velocity vector $\underset{\sim}{v}$.

2 marks

b. Find its maximum speed.

3 marks

c. Show that the particle never stops.

1 mark

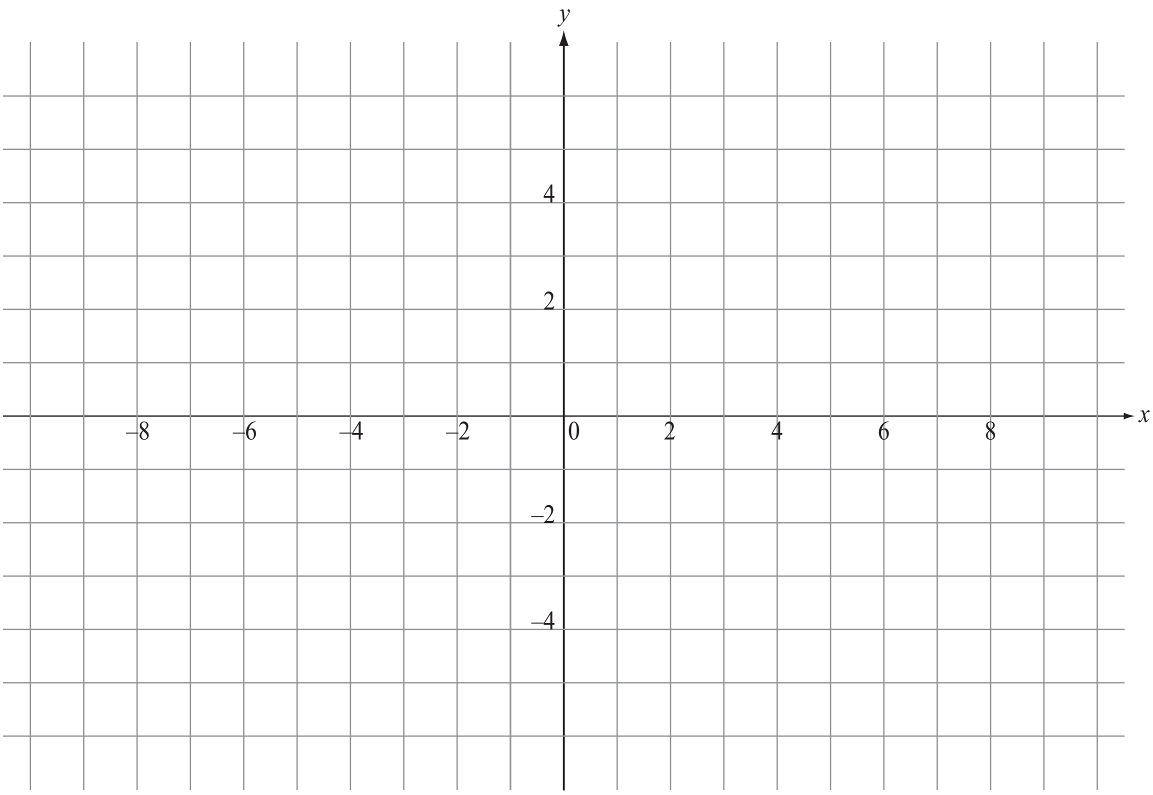
Question 8

The position vector of a particle is given by $\underset{\sim}{r}(t) = 2\tan(t)\underset{\sim}{i} + \sec(t)\underset{\sim}{j}$ where $t \geq 0$.

a. Find the Cartesian equation of the path of the particle.

2 marks

b. Sketch the curve on the grid below, showing all important features.



2 marks

c. Find the exact volume of revolution formed by rotating this curve between $y = 1$ and $y = 2$ about the y-axis.

2 marks

Total 40 marks

END OF QUESTION AND ANSWER BOOK

Specialist Mathematics Exam 1 2007 Solutions

Question 1

The equation has real coefficients therefore the conjugate root theorem applies.
So $2 - i$ is another root. **A1**

The two factors can be expressed as a quadratic as follows:

$(z - 2 - i)(z - 2 + i) = z^2 - 4z + 5$ **A1**

Divide $z^2 - 4z + 5$ into $z^4 - 4z^3 + 6z^2 - 4z + 5$ to obtain $z^2 + 1$ **M1**

$$\begin{array}{r|l} & z^2 + 1 \\ z^2 - 4z + 5 & z^4 - 4z^3 + 6z^2 - 4z + 5 \\ & \underline{z^4 - 4z^3 + 5z^2} \\ & z^2 - 4z + 5 \\ & \underline{z^2 - 4z + 5} \end{array}$$

$\therefore (z^2 - 4z + 5)(z^2 + 1) = 0$

$(z - 2 - i)(z - 2 + i)(z - i)(z + i) = 0$

$\therefore z = 2 + i, 2 - i, i, -i$

Solutions are: $z = 2 \pm i$ and $z = \pm i$ **A1**

Question 2

a. $\underset{\sim}{u} = \cos(\theta)\underset{\sim}{i} + \sin(\theta)\underset{\sim}{j}$ and $\underset{\sim}{v} = \sin(\theta)\underset{\sim}{i} + \cos(\theta)\underset{\sim}{j}$

$$\begin{aligned} |\underset{\sim}{u}| &= \sqrt{\cos^2(\theta) + \sin^2(\theta)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$ **A1**

$$\begin{aligned} |\underset{\sim}{v}| &= \sqrt{\sin^2(\theta) + \cos^2(\theta)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

Hence, both $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are unit vectors.

b. $\cos(\alpha) = \dfrac{\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta)}{\sqrt{1} \times \sqrt{1}}$ **M1**

$$\begin{aligned} &= 2\sin(\theta)\cos(\theta) \\ &= \sin(2\theta) \end{aligned}$$

$\alpha = \cos^{-1}(\sin(2\theta))$ or $\alpha = \dfrac{\pi}{2} - 2\theta$ **A1**

c. $\alpha = \cos^{-1}\left(\sin\left(\dfrac{2 \times \pi}{6}\right)\right)]$ **A1**

$$\begin{aligned} &= \cos^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) \\ &= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\pi}{6} \end{aligned}$$

d. $(\underset{\sim}{v} \cdot \hat{\underset{\sim}{u}})\hat{\underset{\sim}{u}} = \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\right)\left(\frac{1}{2}\underset{\sim}{i} + \frac{\sqrt{3}}{2}\underset{\sim}{j}\right)$ **M1**

$= \frac{\sqrt{3}}{4}\underset{\sim}{i} + \frac{3}{4}\underset{\sim}{j}$ or

$= \frac{1}{4}\left(\sqrt{3}\,\underset{\sim}{i} + 3\underset{\sim}{j}\right)$ **A1**

Question 3

a. $\frac{x+2}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$ where A and B are constants. **A1**

$\therefore x + 2 = A(x+1) + B(x)$

Let $x = 0$ so $A = 2$

Let $x = -1$ so $B = -1$ **A1** (both A and B correct)

$\therefore \frac{x+2}{x^2+u} = \frac{2}{x} - \frac{1}{x+1}$

b. $\int_{-4}^{-3}\left(\frac{x+2}{x^2+x}\right)dx = \int_{-4}^{-3}\left(\frac{2}{x} - \frac{1}{x+1}\right)dx$

$= \left[2\log_e|x| - \log_e|x+1|\right]_{-4}^{-3}$ **A2** for anti-derivatives

$= (2\log_e 3 - \log_e 2) - (2\log_e 4 - \log_e 3)$ Modulus sign missing = –1

$= \log_e(27/32)$

Answer: $a = 27$, $b = 12$

Note: cannot get this mark from logs of negative numbers. Equivalent multiples of *a* and *b* in non-simplified fraction is correct.

Question 4

a. Let $u = \sqrt{3x}$ and $w = 3x$

$u = \sqrt{w}$ and so $\frac{du}{dw} = \frac{1}{2\sqrt{w}}$ and $\frac{dw}{dx} = 3$

$\frac{du}{dx} = \frac{du}{dw} \times \frac{dw}{dx}$

$= \frac{3}{2\sqrt{3x}}$ **A1**

$y = \cos^{-1}(u)\ and\ so\ \frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{-1}{\sqrt{1-u^2}} \times \frac{3}{2\sqrt{3x}}$ **M1**

$= \frac{-1}{\sqrt{1-3x}} \times \frac{3}{2\sqrt{3x}}$

$= \frac{-3}{2\sqrt{3x(1-3x)}}$

Hence shown.

b. $\displaystyle\int_{\frac{1}{12}}^{\frac{1}{6}} \frac{1}{\sqrt{3x(1-3x)}}dx$ **A1** for $-\frac{2}{3}$ in front

$= -\dfrac{2}{3}\displaystyle\int_{\frac{1}{12}}^{\frac{1}{6}} \frac{-3}{2\sqrt{3x(1-3x)}}dx$ **M1** for recognition

$= -\frac{2}{3}\left[\cos^{-1}\left(\sqrt{3x}\right)\right]_{\frac{1}{12}}^{\frac{1}{6}}$

$= -\frac{2}{3}\left(\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) - \cos^{-1}\left(\frac{1}{2}\right)\right)$

$= -\frac{2}{3}\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$

$= \frac{\pi}{18}$ **A1**

Question 5

a. $2a = 2g - 0.05v^2 \ \therefore a = g - \frac{v^2}{40}$ **A1**

b. Using $a = v\frac{dv}{dx}$ in the equation of motion gives:

$v\frac{dv}{dx} = \frac{2g - 0.05v^2}{2}$ **M1**

$\frac{dv}{dx} = \frac{2g - 0.05v^2}{2v}$

$\frac{dx}{dv} = \frac{2v}{2g - 0.05v^2}$ **A1**

Multiplying numerator and denominator by 20 gives

$\frac{dx}{dv} = \frac{40v}{40g - v^2}$ as required.

c. The required distance is given by the integral: $\displaystyle\int_0^{10} \frac{40v}{40g - v^2}dv$ **A1**

Note: The integral must have correct limits and dv. Does not need to have a modulus of $\frac{40v}{40g - v^2}$, since we are after distance and the graph was not asked for.

$x = -20\displaystyle\int_0^{10} \frac{-2v}{-v^2 + 40g}dv$ **M1**

$= \left[-20\log_e\left(40g - v^2\right)\right]_0^{10}$ **M1**

$= -20\log_e(40g - 100) + 20\log_e(40g)$

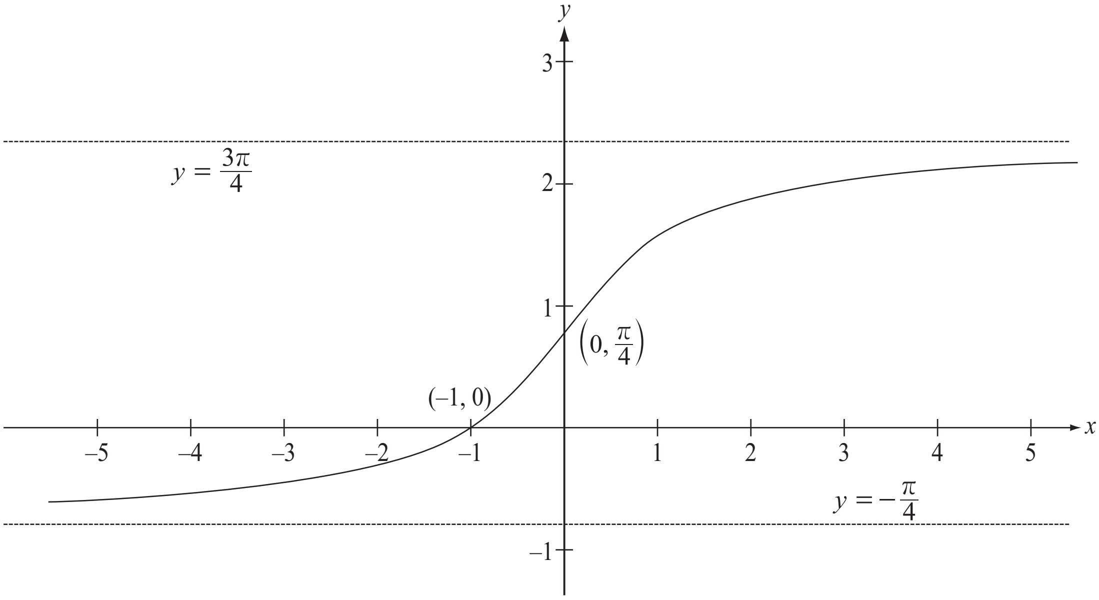
$= 20\log_e\left(\frac{40g}{40g - 100}\right)$

$= 20\log_e\left(\frac{2g}{2g - 5}\right)$

Note: $20\log_e\left(\frac{40g}{40g - 100}\right)$ can get the last A1 mark. **A1**

Question 6

a.



x-intercept $(-1, 0)$ **A1**

y-intercept $\left(0, \frac{\pi}{4}\right)$ **A1**

Asymptotes $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$ and shape. **A1**

b. $\arctan(x) + \frac{\pi}{4} = \frac{5\pi}{12}$

$\arctan(x) = \frac{\pi}{6}$

$x = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ **A1**

Question 7

a. Differentiating $\underset{\sim}{r}$ with respect to t:

$\underset{\sim}{r} = (-3\sin(t) - 2\cos(2t))\underset{\sim}{i} + (3\cos(t) - 2\sin(2t))\underset{\sim}{j}$ **A2** (1 each $\underset{\sim}{i}$, $\underset{\sim}{j}$ term)

b. Speed $= |\underset{\sim}{v}|$

$= \sqrt{(3\sin(t) + 2\cos(2t))^2 + (3\cos(t) - 2\sin(2t))^2}$ **M1**

$= \sqrt{9\sin^2(t) + 12\sin(t)\cos(2t) + 4\cos^2(2t) + 9\cos^2(t) - 12\cos(t)\sin(2t) + 4\sin^2(2t)}$

$= \sqrt{(9\sin^2(t) + 9\cos^2(t)) + 12(\sin(t)\cos(2t) - 12\cos(t)\sin(2t)) + (4\cos^2(2t) + 4\sin^2(2t))}$

$= \sqrt{9 + 4 + 12\sin(t - 2t)}$ **M1** for using the compound angle formula

$= \sqrt{13 - 12\sin(t)}$

$\therefore$ Maximum speed is $\sqrt{13 + 12}$ when $\sin(t) = -1$

$\therefore$ Maximum speed is 5. **A1**

c. $\sqrt{13 - 12\sin(t)}$

$-1 \le \sin(t) \le 1$

$\therefore -12 \le 12\sin(t) \le 12$

$\therefore \sqrt{13-12} = 1, \sqrt{13+12} = 5$

$\therefore$ speed will always be between 1 and 5 **A1**

$\therefore$ it never stops

Question 8

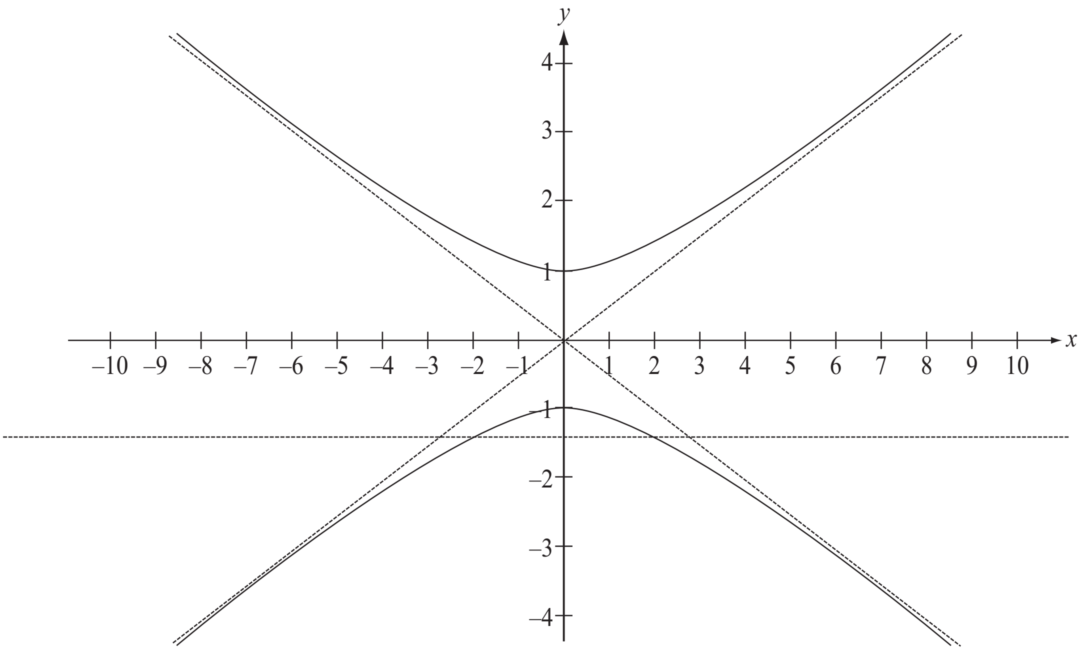
a. $\frac{x}{2} = \tan(t)$ and $y = \sec(t)$

$1 + \tan^2(t) = \sec^2(t)$ **M1**

$1 + \frac{x^2}{4} = y^2$

$1 = \frac{y^2}{1} - \frac{x^2}{4}$

b.



2 marks: A1 shape and asymptotes $y = \pm\frac{x}{2}$; **A1** y-intercepts $(0, \pm 1)$

c. $\int_1^2 \pi x^2 \, dy = \int_1^2 4\pi(y^2 - 1) \, dy$

$= \left[4\pi\left(\frac{y^3}{3} - y\right)\right]_1^2$ **M1**

$= 4\pi\left[\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)\right]$

$= \frac{16\pi}{3}$ cubic units **A1**

SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.

Version 3 – March 2006

Specialist Mathematics Formulas

Mensuration

area of a trapezium: $\frac{1}{2}(a+b)h$

curved surface area of a cylinder: $2\pi rh$

volume of a cylinder: $\pi r^2 h$

volume of a cone: $\frac{1}{3}\pi r^2 h$

volume of a pyramid: $\frac{1}{3}Ah$

volume of a sphere: $\frac{4}{3}\pi r^3$

area of a triangle: $\frac{1}{2}bc\sin A$

sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$

Coordinate geometry

ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

hyperbola: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Circular (trigometric) functions

$\cos^2(x) + \sin^2(x) = 1$

$1 + \tan^2(x) = \sec^2(x)$

$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$

$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$

$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$\sin(2x) = 2\sin(x)\cos(x)$

$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$

function	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
domain	$[-1, 1]$	$[-1, 1]$	R
range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Algebra (Complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r \operatorname{cis}\theta$

$|z| = \sqrt{x^2 + y^2} = r$

$-\pi < \operatorname{Arg} z \le \pi$

$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$

$z^n = r^n \operatorname{cis}(n\theta)$ (de Moivre's theorem)

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ $\quad\quad$ $\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$ $\quad\quad$ $\int e^{ax} dx = \frac{1}{a} e^{ax} + c$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$ $\quad\quad$ $\int \frac{1}{x} dx = \log_e |x| + c$

$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$ $\quad\quad$ $\int \sin(ax)\, dx = -\frac{1}{a}\cos(ax) + c$

$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$ $\quad\quad$ $\int \cos(ax)\, dx = \frac{1}{a}\sin(ax) + c$

$\frac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$ $\quad\quad$ $\int \sec^2(ax)\, dx = \frac{1}{a}\tan(ax) + c$

$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$ $\quad\quad$ $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c, a > 0$

$\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$ $\quad\quad$ $\int \frac{-1}{\sqrt{a^2-x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c, a > 0$

$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$ $\quad\quad$ $\int \frac{a}{a^2+x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Euler's method: If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$

acceleration: $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant (uniform) acceleration: $v = u + at$ $\quad$ $s = ut + \frac{1}{2}at^2$ $\quad$ $v^2 = u^2 + 2as$ $\quad$ $s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$|\underset{\sim}{r}| = \sqrt{x^2 + y^2 + z^2} = r$

$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2$

$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$

Mechanics

momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$

equation of motion: $\underset{\sim}{R} = m\underset{\sim}{a}$

friction: $F \le \mu N$

END OF FORMULA SHEET