THE SCHOOL FOR EXCELLENCE

UNIT 4 SPECIALIST MATHEMATICS 2006

COMPLIMENTARY WRITTEN EXAMINATION 1 - SOLUTIONS

QUESTION 1

a. $\overrightarrow{CD} = \overrightarrow{CO} + \overrightarrow{OD} = -\overrightarrow{OC} + \overrightarrow{OD}$.

$\overrightarrow{OC} = 5\underset{\sim}{i} + 5\underset{\sim}{j}$.

Since *D* is the midpoint of *AB*, it has coordinates (2, 1). Therefore $\overrightarrow{OD} = 2\underset{\sim}{i} + \underset{\sim}{j}$.

Therefore $\overrightarrow{CD} = -5\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{i} + \underset{\sim}{j} = -3\underset{\sim}{i} - 4\underset{\sim}{j}$.

b. $\underset{\sim}{a} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$ $\quad \underset{\sim}{b} = 6\underset{\sim}{i} - 2\underset{\sim}{j}$ $\quad \underset{\sim}{c} = 5\underset{\sim}{i} + 5\underset{\sim}{j}$

$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = 8\underset{\sim}{i} - 6\underset{\sim}{j}$

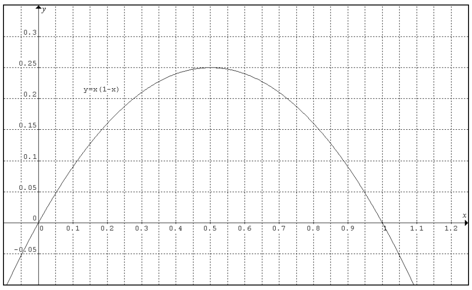
$\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$

$\overrightarrow{CD} = -\underset{\sim}{c} + \underset{\sim}{a} + \overrightarrow{AD} = -3\underset{\sim}{i} - 4\underset{\sim}{j}$

$\overrightarrow{AB} \bullet \overrightarrow{CD} = (8 \times -3 + -6 \times -4) = (-24 + 24) = 0$

Hence $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CD}$.

QUESTION 2



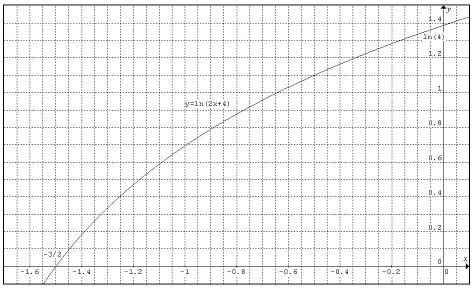
$$\delta v = \pi y^2 \delta x$$

$$Vol = \int_0^1 \left(\pi y^2\right) dx = \pi \int_0^1 \left(x^2 - 2x^3 + x^4\right) dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right]_0^1$$

$$= \pi \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right] - \pi[0] \ = \frac{\pi}{30} \text{ cubic units}$$

QUESTION 3



a.

$$y = \log_e(2x+4)$$

$$x = \log_e(2y+4)$$

$$e^x = 2y+4$$

$$y = \frac{e^x}{2} - 2$$

$$f^{-1} : R \to R,\ f^{-1}(x) = \frac{e^x}{2} - 2$$

$$Range(f) = R \Rightarrow Dom\left(f^{-1}\right) = R$$

b. **Y Intercept:** $f(0) = \log_e 4$

X Intercept: $\log_e(2x+4) = 0$

$$2x+4 = 1$$

$$x = \frac{-3}{2}$$

$$Area = \int_{\frac{-3}{2}}^{0} \log_e(2x+4)\, dx$$

c. $y = \log_e(2x+4) \Rightarrow x = \dfrac{e^y}{2} - 2$

$$\int_{\frac{-3}{2}}^{0} y\ dx = -\int_0^{\log_e 4} x\ dy = \int_{\log_e 4}^{0} \left(\frac{e^y}{2} - 2\right) dy$$

$$= \left[\frac{e^y}{2} - 2y\right]_{\log_e 4}^{0} = \left[\frac{1}{2}\right] - \left[\frac{4}{2} - 2\log_e 4\right]$$

$$= 2\log_e 4 - \frac{3}{2} \text{ square units}$$

QUESTION 4

a. $x = 4\sin^3(t)$

$y = \cos(2t)$

b. $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

$$\frac{dy}{dt} = -2\sin(2t)$$

$$\frac{dx}{dt} = 12\sin^2(t)\cos(t)$$

$$\frac{dt}{dx} = \frac{1}{12\sin^2(t)\cos(t)}$$

$$\frac{dy}{dx} = \frac{-2\sin(2t)}{12\sin^2(t)\cos(t)} = \frac{-2\sin(2t)}{6\sin(t)\times 2\sin(t)\cos(t)} = \frac{-2\sin(2t)}{6\sin(t)\sin(2t)}$$

$$\frac{dy}{dx} = \frac{-1}{3\sin(t)}$$

c. $x\left(\dfrac{\pi}{6}\right) = 4\sin^3\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$

$$y\left(\frac{\pi}{6}\right) = \cos\left(2\times\frac{\pi}{6}\right) = \frac{1}{2}$$

$$m(normal) = -\frac{dx}{dy} = 3\sin\left(\frac{\pi}{6}\right) = \frac{3}{2}$$

Equation: Use $(y - y_1) = m(x - x_1)$

$$\left(y - \frac{1}{2}\right) = \frac{3}{2}\left(x - \frac{1}{2}\right)$$

$$y = \frac{3x}{2} - \frac{1}{4}$$

QUESTION 5

a. (i) $\dfrac{dh}{dt} = k,\ k \in R$

$\therefore h = kt + c$

Substitute $(0, 10)$: $c = 10$

$\therefore h = kt + 10$

Substitute $(5, 40)$:

$5 = 40k + 10$

$k = -\dfrac{1}{8}$

$\therefore h = -\dfrac{t}{8} + 10$

(ii) Substitute $t = 60$: $h = -\dfrac{60}{8} + 10 = -\dfrac{15}{2} + 10 = \dfrac{5}{2}$ metres.

b. (i) $\dfrac{dh}{dt} = -kh,\ \ k \in R^{+}$

$\dfrac{dt}{dh} = -\dfrac{1}{kh}$

$t = -\dfrac{1}{k}\int\left(\dfrac{1}{h}\right) dh = -\dfrac{1}{k}\log_e h + c$

$\therefore h = Ae^{-kt}$ where $A = e^{kc}$

Substitute $(0, 10)$**:** $A = 10$

$h = 10e^{-kt}$

Substitute $(5, 40)$**:** $5 = 10e^{-40k}$

$-40k = \log_e\left(\dfrac{1}{2}\right)$

$k = \dfrac{1}{40}\log_e 2 = \log_e 2^{\frac{1}{40}}$

Therefore: $h = 10e^{-t\log_e 2^{\frac{1}{40}}}$

$h = 10e^{-kt}$ where $k = \dfrac{1}{40}\log_e 2$.

b. (ii) From (i): $h = 10e^{-kt}$ where $k = \frac{1}{40}\log_e 2$.

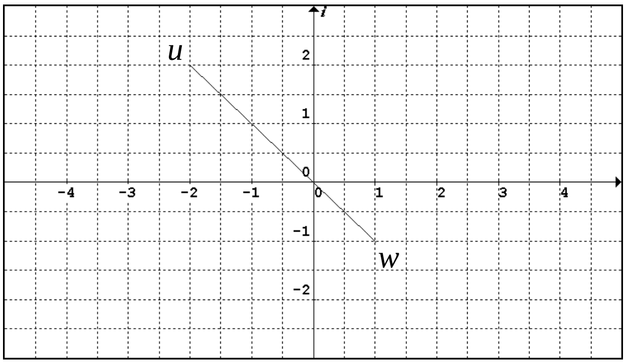
Substitute $t = 60$: $h = 10e^{-60k}$ where $-60k = -\frac{60}{40}\log_e 2 = -\frac{3}{2}\log_e 2 = \log_e 2^{-3/2}$.

Therefore $h = 10e^{\log_e 2^{-3/2}} = (10)(2^{-3/2}) = \frac{10}{2^{3/2}} = \frac{10}{2\sqrt{2}} = \frac{5}{\sqrt{2}}$ metres.

QUESTION 6

a. $w = \frac{4i}{u} = \frac{4i}{-2+2i} \times \frac{-2-2i}{-2-2i} = \frac{-8i+8}{8} = 1-i$

b.



From Argand Diagram:

$|u| = \sqrt{4+4} = 2\sqrt{2}$

$Arg(u) = \frac{3\pi}{4}$

$u = 2\sqrt{2}cis\left(\frac{3\pi}{4}\right)$

$|w| = \sqrt{1+1} = \sqrt{2}$

$Arg(w) = -\frac{\pi}{4}$

$w = \sqrt{2}cis\left(-\frac{\pi}{4}\right)$

c. $4i = uw$

$\sqrt{2}cis\left(\frac{3\pi}{4}\right) = \frac{u}{2}$

Hence: $ua + 4i = \sqrt{2}cis\left(\frac{3\pi}{4}\right)$

$ua + uw = \frac{u}{2}$

$u(a+w) = \frac{u}{2}$

$a + w = \frac{1}{2}$ $\quad a = \frac{1}{2} - w = -\frac{1}{2} + i$

QUESTION 7

Note: $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$.

a. Examine particle B

$$\underset{\sim}{F_N} = 3mg - T = 3ma$$

$$3mg - T = \frac{3mg}{2}$$

$$T = 3mg - \frac{3mg}{2}$$

$$T = \frac{3mg}{2}\,Newtons$$

b. Examine particle A

Perpendicular to the plane:

$$N = mg\cos\alpha = \frac{4mg}{5}$$

Parallel to the plane:

$$T - \mu N - mg\sin\alpha = ma$$

$$T - \frac{4\mu mg}{5} - \frac{3mg}{5} = \frac{mg}{2}$$

$$\frac{4\mu}{5} = \frac{3}{2} - \frac{3}{5} - \frac{1}{2} = \frac{2}{5}$$

$$\mu = \frac{1}{2}$$

QUESTION 8

$x = \cos t$

When $x = 0$: $t = \cos^{-1}(0) = \frac{\pi}{2}$

$\frac{dx}{dt} = -\sin t$

When $x = \frac{1}{2}$: $t = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$\frac{dt}{dx} = \frac{-1}{\sin t}$

$$\frac{x^2}{\sqrt{1-x^2}} = \frac{\cos^2 t}{\sqrt{1-\cos^2 t}} = \frac{\cos^2 t}{\sqrt{\sin^2 t}} = \frac{\cos^2 t}{\sin t} = \cos^2 t \times -\frac{dt}{dx}$$

Hence: $\int_0^{\frac{1}{2}} \frac{x^2}{\left(1-x^2\right)^{\frac{1}{2}}}\, dx = -\int_{t=\frac{\pi}{2}}^{t=\frac{\pi}{3}} \cos^2 t \, \frac{dt}{dx} dx$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2 t \, dt = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos(2t) + 1 \, dt$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin(2t) + t\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{1}{2}\left[\frac{1}{2}\sin\pi + \frac{\pi}{2} - \frac{1}{2}\sin\frac{2\pi}{3} - \frac{\pi}{3}\right]$$

$$= \frac{1}{2}\left[0 + \frac{3\pi}{6} - \frac{\sqrt{3}}{4} - \frac{2\pi}{6}\right]$$

$$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$