THE SCHOOL FOR EXCELLENCE (TSFX)
UNIT 4 SPECIALIST MATHEMATICS 2006

WRITTEN EXAMINATION 2

Reading Time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOKLET

Structure of Booklet

Section	Number of questions	Number of questions to be answered	Number of marks
1 2	22 4	22 4	22 58 **Total 80**

This examination has two sections: Section 1 (multiple-choice questions) and Section 2 (extended-answer questions).

You must complete both parts in the time allocated. When you have completed one part continue immediately to the other part.

Students are permitted to bring into the examination rooms: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved **graphics** calculator or CAS calculator or CAS software (memory **DOES NOT** need to be cleared) and, if desired, one scientific calculator.

Students are **NOT** permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Students are **NOT** permitted to bring mobile phones and/or any electronic communication devices into the examination room.

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SECTION 1 – MULTIPLE CHOICE QUESTIONS

Instructions for Section 1

Answer all questions in this part on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

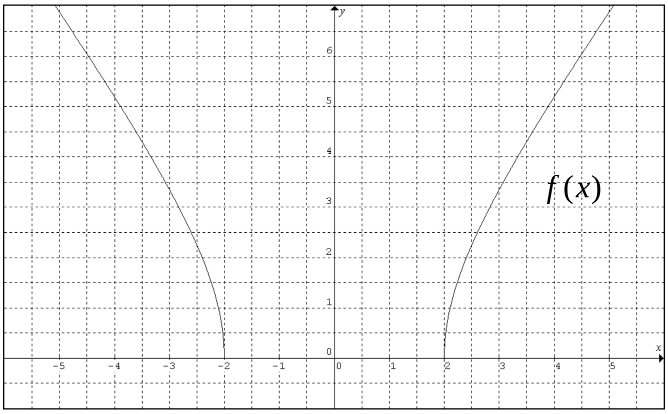
A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers. You should attempt every question.

No marks will be given if more than one answer is completed for any question.

Take the acceleration due to gravity to have magnitude $g\ m/s^2$, where $g = 9.8$.

The following information relates to Questions 1 and 2



QUESTION 1

If $f(4) = \frac{3\sqrt{12}}{2}$, then the relation describing $f(x)$ and the equations of the asymptotes are:

A $\quad \frac{x^2}{9} - \frac{y^2}{4} = 1$ and $y = \pm\frac{3x}{2}$

B $\quad \frac{x^2}{9} - \frac{y^2}{4} = 1$ and $y = \pm\frac{2x}{3}$

C $\quad \frac{x^2}{4} - \frac{y^2}{9} = 1$ and $y = \pm\frac{2x}{3}$

D $\quad \frac{x^2}{4} - \frac{y^2}{9} = 1$ and $y = \pm\frac{3x}{2}$

E $\quad \frac{x^2}{9} + \frac{y^2}{4} = 1$ and $y = \pm\frac{3x}{2}$

QUESTION 2

The equations of the asymptotes for the graph of the function $f(ax+b)$ are:

A $y = ax + b$ and $y = -ax + b$

B $y = \frac{3ax}{2} + \frac{3b}{2}$ and $y = \frac{-3ax}{2} - \frac{3b}{2}$

C $y = \frac{2ax}{3} + \frac{2b}{3}$ and $y = \frac{-2ax}{3} - \frac{2b}{3}$

D $y = \frac{2bx}{3} + \frac{2a}{3}$ and $y = \frac{-2bx}{3} - \frac{2a}{3}$

E $y = \frac{3bx}{2} + \frac{3a}{2}$ and $y = \frac{-3bx}{2} - \frac{3a}{2}$

QUESTION 3

The gradient of the curve $ax + by^2 = 1$ at the point where $y = c$ is:

A $\dfrac{-a}{2b\sqrt{\dfrac{(1-ax)}{b}}}$

B $\dfrac{-a}{2b\sqrt{\dfrac{(1-ac)}{b}}}$

C $\frac{1}{2bc} - \frac{a}{2bc}$

D $\frac{-2bc}{a}$

E $\frac{-a}{2bc}$

QUESTION 4

If one solution to the equation $z^2 = a, a \in c$ is $z_1 = m + ni$ and z_2 is the other solution, then $|a| - |z_2|$ is:

A $\sqrt{m^2 + n^2}\left(\sqrt{m^2 + n^2} - 1\right)$

B $\sqrt{\left(m^2 - n^2 + m\right)^2 + \left(2mn - n\right)^2}$

C $\sqrt{\left(m^2 - n^2 - m\right)^2 + \left(2mn - n\right)^2}$

D $\sqrt{\left(m^2 - n^2\right)^2 + \left(2mn\right)^2}$

E None of the above

QUESTION 5

If $z = 1 + \sqrt{3}i$ is a solution to the equation $z^3 - a = 0, a \in c$ then the other two solutions are:

A 2 and 0

B 2 and $1 - \sqrt{3}i$

C 2 and $\sqrt{3} - i$

D –2 and $1 - \sqrt{3}i$

E –2 and $\sqrt{3} - i$

The following information relates to Questions 6 and 7

The position vector describing the path of a particle is given by:

$$r(t) = a\cos(t)\underset{\sim}{i} + 3\cos(t)\underset{\sim}{j}, t \geq 0$$

where distances are in metres and time is in seconds.

QUESTION 6

The value of a such that at $t = \frac{\pi}{4}$ seconds, the particle's speed is $\sqrt{17}\ ms^{-1}$ and its direction of motion is greater than $\frac{\pi}{2}$ from the positive x axis is equal to

A $a = \sqrt{34} - 3$
B $a = -5$
C $a = 5$
D $a = \sqrt{43}$
E $a = -\sqrt{43}$

QUESTION 7

A unit vector parallel to $\dot{r}\left(\frac{\pi}{3}\right)$ is

A $\frac{1}{\sqrt{a^2+9}}\left(a\underset{\sim}{i} - 3\underset{\sim}{j}\right)$

B $\frac{1}{\sqrt{a^2+9}}\left(-a\underset{\sim}{i} - 3\underset{\sim}{j}\right)$

C $\frac{1}{\sqrt{a^2+9}}\left(-a\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

D $\frac{1}{\sqrt{a^2+9}}\left(a\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

E $\frac{1}{\sqrt{a^2-9}}\left(a\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

QUESTION 8

If $f(x)$ is positive and increasing on $0 \le x \le b$ with $f(0) = a$, $f(b) = 5$ and $\int_a^5 f^{-1}(x)\,dx = b + a$, then the definite integral $\int_0^b f(x)\,dx$ is equal to:

A $a - 4b$
B $b - 4a$
C $4a - b$
D $4b - a$
E $5a(a + b)$

QUESTION 9

The velocity, v ms^{-1}, at time t s, of a body moving in a straight line is given by the equation $v = t^2 - t - 2$, $t \ge 0$. Its acceleration when the velocity is zero is:

A Zero
B 3 ms^{-2}
C 5 ms^{-2}
D –2.5 ms^{-2}
E –3 ms^{-2}

QUESTION 10

The position of a particle at time t is given by $\underset{\sim}{r}(t) = 3\cos(3t)\underset{\sim}{i} + 3\sin(3t)\underset{\sim}{j}$, $t \ge 0$.

The cartesian equation of the particle's motion is:

A $x^2 + y^2 = 1$
B $3x^2 + 3y^2 = 1$
C $y = 3\sin(3t), \ t \ge 0$
D $y = 3(\cos(3t) + \sin(3t)), \ t \ge 0$
E $x^2 + y^2 = 9$

QUESTION 11

A particle moves in a straight line and, at time t, its displacement from a fixed origin is x.

If $\ddot{x} = \dfrac{1}{(t+1)^3}$ and $\dot{x} = x = 1$ when $t = 0$, the value of x when $t = 2$ is:

A $\dfrac{1}{6}$

B 1

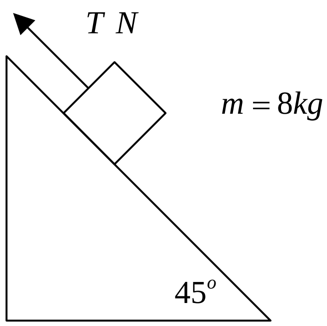
C $\dfrac{8}{6}$

D $\dfrac{16}{6}$

E $\dfrac{22}{6}$

QUESTION 12

A particle of mass 8.00 kg sits on a rough plane inclined at 45° to the horizontal. The coefficient of friction between the particle and the plane is 0.2. The particle is supported by a rope exerting a force of T Newtons up the plane and parallel to the plane. The force of T Newtons **just prevents the particle from sliding down the plane**. The magnitude of the force T Newtons is closest to:



A 39.76 N
B 44.35 N
C 62.72 N
D 66.53 N
E 71.12 N

QUESTION 13
A hot liquid in a cup left to cool obeys Newton's law of cooling. The rate of decrease of temperature (T) is proportional to the difference between the temperature of the liquid and the ambient room temperature $\left(T_0 = 25^\circ C\right)$.

When the liquid in a particular cup is at $50^\circ C$ the rate of cooling is $0.5^\circ C$ per minute without any external heating. When the liquid reaches room temperature $\left(T_0 = 25^\circ C\right)$, it is placed on a hotplate that heats it at a rate of $1.5^\circ C$ per minute, independent of any cooling effects.

The differential equation describing the rate of change of temperature with time after the liquid is placed on the hotplate is:

A $-\frac{1}{50}(T-25)$

B $-1.5(T-25)$

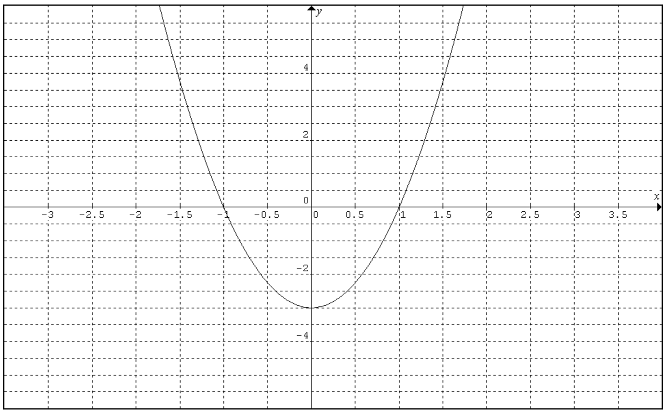
C $1.5-\frac{1}{50}(T-25)$

D $1.5-1.5(T-25)$

E $50-\frac{1}{50}(T-25)$

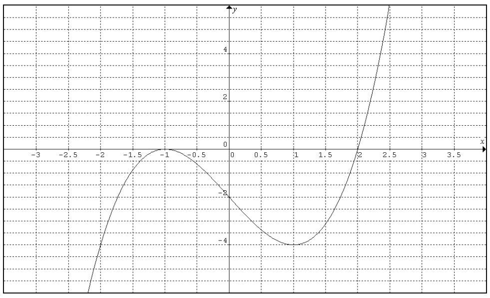
QUESTION 14

The graph of the quadratic function $y = f(x)$ is shown below.

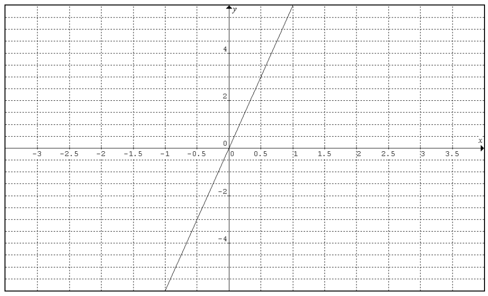


Given that $F\left(\frac{3}{2}\right) = -3.125$ the graph of $F(x)$, the antiderivative of $f(x)$ is:

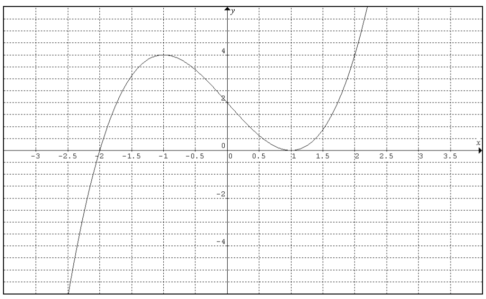
A



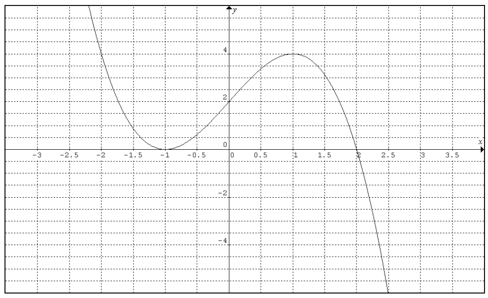
D



B



E



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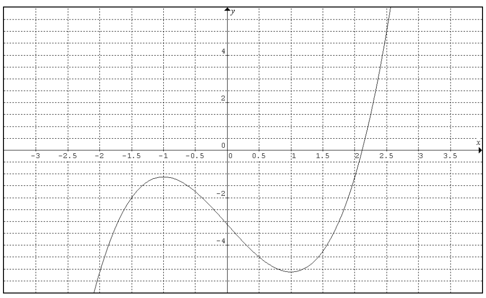


QUESTION 15

If $f(x)=(x-2)^2(x+1)$, a graph of the antiderivative of $f(x)$ has:

A A maximum turning point at $x=-1$and a point of inflection at $x=2$.
B A minimum turning point at $x=-1$and a point of inflection at $x=2$.
C A point of inflection at $x=1$ and a minimum turning point at $x=-2$.
D Minimum turning points at $x=0$ and $x=2$.
E Points of inflection at $x=0$ and $x=2$ and a minimum turning point at $x=-1$.

QUESTION 16

If $\sin(x)=a$ and $\sec(x)=-b$, $a,b>0$ then $\cot(-x)$ is equal to:

A $\frac{1}{ab}$

B $-\frac{1}{ab}$

C ab

D $\frac{a}{b}$

E $-\frac{a}{b}$

QUESTION 17

$y=\frac{1}{2}xe^{3x}$ is a solution to:

A $3\frac{dy}{dx}-\frac{d^2y}{dx^2}-3y=0$

B $3\frac{dy}{dx}-\frac{d^2y}{dx^2}+3y=0$

C $3\frac{d^2y}{dx^2}-\frac{dy}{dx}+3y=0$

D $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+3y=0$

E $\frac{d^2y}{dx^2}+\frac{dy}{dx}=12y$

QUESTION 18

With a suitable substitution, $\int \frac{e^{3x}}{e^{2x}+1}\,dx$ can be written as:

A $\int\left(1-\frac{1}{u^2+1}\right)du$

B $\int\left(\frac{u^3}{u^2+1}\right)du$

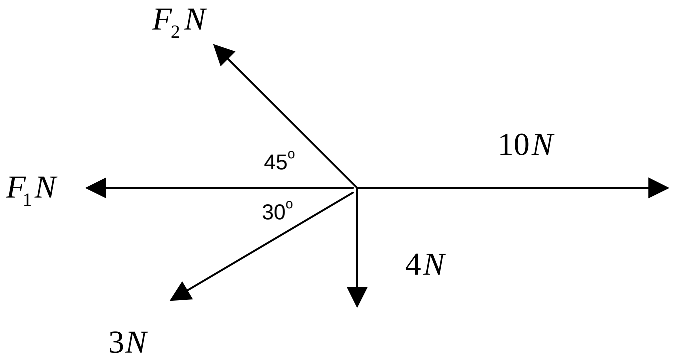
C $\frac{1}{2}\int\left(\frac{u-1}{u}\right)du$

D $\frac{1}{2}\int\left(\frac{u}{u^2+1}\right)du$

E None of the above.

QUESTION 19

The diagram shows a system of forces acting on a particle:



The values of F_1 and F_2 respectively are closest to:

A $F_1 = 4.90N$ and $F_2 = 3.54N$

B $F_1 = 1.90N$ and $F_2 = 7.78N$

C $F_1 = 3.93N$ and $F_2 = 10.47N$

D $F_1 = 7.40N$ and $F_2 = 10.89N$

E $F_1 = 10.89N$ and $F_2 = 7.40N$

QUESTION 20

If $f'(x) = x^{\frac{1}{3}}$ and $f(8) = 3$, using Euler's method with step size of 1, $f(11)$ is closest to:

A 9.000
B 9.160
C 9.234
D 12.000
E 4.158

QUESTION 21

If $u = -\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} i$ and $v = 1 - \sqrt{3}i$ then $u^2 v$ in polar form, expressed with principal Argument is:

A $6cis\left(\frac{5\pi}{12}\right)$

B $9cis\left(\frac{9\pi}{16}\right)$

C $18cis\left(\frac{7\pi}{6}\right)$

D $18cis\left(\frac{-5\pi}{6}\right)$

E $18cis\left(\frac{11\pi}{48}\right)$

QUESTION 22

The range of the function f with the rule $f(x) = b + a\cos^{-1}(cx)$ is:

A $[0, \pi]$

B $\left[-\frac{1}{c}, \frac{1}{c}\right]$

C $[0, a\pi]$

D $[0, a\pi + b]$

E $[b, a\pi + b]$

SECTION 2 – EXTENDED ANSWER QUESTIONS

Instructions For Section 2

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Take the acceleration due to gravity to have magnitude $g\ m/s^2$, where $g = 9.8$.

QUESTION 1

A hot cup of coffee left on the teacher's desk in a Specialist Maths Class cools according to Newton's Law of Cooling. It can be shown that the temperature of the coffee, $T^\circ C$, varies with the time, t minutes, according to the equation:

$$T = Ae^{-kt} + T_r, \quad k > 0$$

where T_r is the ambient room temperature and A and k represent real number constants. Assume that T_r remains constant at $20^\circ C$ throughout this question.

a. (i) If $T(0) = T_0$, find A in terms of T_0.

1 mark

(ii) For black coffee (without milk), $T_0 = 90^\circ C$. If the coffee has cooled to $80^\circ C$ in 5 minutes, find A and k, correct to 5 decimal places.

1 mark

(iii) Hence find the time it takes for the coffee to cool to $55^{\circ}C$, correct to the nearest one-tenth of a second.

1 mark

Adding milk to black coffee changes the rate of cooling by affecting the value of k. As more milk is added, k tends to decrease.

When two volumes v_1 and v_2 of liquids at temperatures T_1 and T_2 respectively are mixed together, the initial temperature of the mixture is equal to $\dfrac{v_1T_1 + v_2T_2}{v_1 + v_2}$.

b. $m\ ml$ of milk at $5^{\circ}C$ is added to $v\ ml$ of black coffee at $90^{\circ}C$ to make a total volume of $250\ ml$.

(i) Find an expression for the initial temperature of the coffee, T_0, and hence find an equation describing the constant A in terms of m.

2 marks

(ii) Find the exact value of A for a cup of coffee which has had $5\,ml$ of milk added.

1 mark

(iii) Black coffee with $5\,ml$ of milk added is found to cool to $73.8^\circ C$ in exactly 10 minutes. Find the value of k, correct to 5 decimal places.

2 marks

c. The relationship between k and m can be modelled by the differential equation

$$\frac{dk}{dm} = a\log_e(m+1), \quad m < 7, \text{ where } a \text{ is a real constant.}$$

Use your values of k for $m = 0$ and $m = 5$ to find the value of a, correct to 5 decimal places.

4 marks

d. (i) Find the derivative of $(x+1)\log_e(x+1)$.

2 marks

(ii) Hence solve the differential equation $\frac{dk}{dm} = a\log_e(m+1)$ for k as a function of m and a.

3 marks

(iii) Hence find the value of a, correct to 6 decimal places.

2 marks

e. A curious Specialist Maths student decides to investigate the time required for the coffee to cool to $55^\circ C$ given different values of m, the amount of milk added.

(i) Solve the equation $T = Ae^{-kt} + 20,\ k > 0$ to find t in terms of $A,\ k$ and T.

1 mark

(ii) Hence write an expression for the time it takes the coffee to cool to $55^\circ C$ in terms of m.

2 marks

Total Marks = 22

QUESTION 2

Let $a = 1 - i$.

a. (i) Show that $z - a$ is a factor of $z^2 - 2z + 2$.

2 marks

(ii) Write down another linear factor of $z^2 - 2z + 2$.

1 mark

(iii) Expand $(z + 1 - i)(z + 1 + i)$.

1 mark

b. Hence factorise $z^4 + 4$ as two quadratic factors over **R.**

2 marks

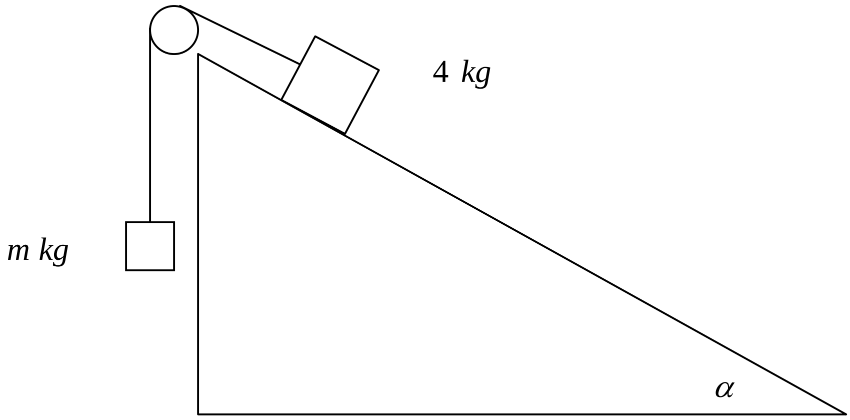
c. Hence solve $z^4 + 4 = 0$ over **C**.

2 marks

Total Marks = 8

QUESTION 3

A particle of mass $4\ kg$ sits on a rough plane inclined at angle α to the horizontal where $\tan\alpha = \frac{3}{4}$. The mass is connected by a light inextensible string, which passes over a light smooth pulley to a mass $m\ kg$, which hangs freely. The coefficient of friction between the $4\ kg$ mass and the plane is μ.



a. If a mass $m = 1.760\ kg$ is just enough to prevent the $4\ kg$ mass from sliding down the plane, find the exact value of μ.

3 marks

b. The mass $m\ kg$ is now replaced by an empty bucket, of mass $0 \cdot 5\ kg$.

(i) Find the acceleration of the $4\ kg$ mass down the plane in terms of g.

2 marks

(ii) If the $4\ kg$ mass starts from rest at the top of the plane, find correct to 3 decimal places its speed and distance traveled after 2 seconds.

2 marks

c. After the system has been in motion for 2 seconds, water is poured into the bucket at a rate of $50\ ml$ per second. Note that $1\ ml$ of water has a mass of $0 \cdot 001\ kg$. Let $t = 0$ at the end of the 2 second period described in Part (b).

(i) Write an equation for the mass of the bucket as a function of time.

1 mark

(ii) Hence write an equation for the acceleration of the $4\ kg$ particle in terms of t.

3 marks

d. (i) Show that $\frac{a-bt}{c+bt}$ can be expressed as $\frac{a+c}{c+bt}-1$.

1 mark

(ii) Hence find an expression for the velocity of the $4\ kg$ particle in terms of t.

3 marks

(iii) Hence find how long it takes for the particle to stop moving, correct to the nearest one-hundredth of a second.

1 mark

(iv) Find how far down the plane the particle travels before it stops moving, correct to the nearest centimeter.

3 marks

Total Marks = 18

QUESTION 4

Let $\overrightarrow{OA} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$ and $\overrightarrow{OB} = x\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$.

a. Write $\overrightarrow{AB}$ in terms of x.

1 mark

b. Find x given that $\left|\overrightarrow{AB}\right| = 3$ and $\left|\overrightarrow{OB}\right| < 6$.

1 mark

Let $\overrightarrow{OC} = \underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$.

c. (i) Write $\overrightarrow{AC}$ and $\left|\overrightarrow{AC}\right|$ in terms of y and z.

1 mark

(ii) If $\overrightarrow{AC}$ is orthogonal to $\overrightarrow{AB}$, write a relationship between y and z.

1 mark

(iii) Hence find the value(s) of y and z that minimize $\left|\overrightarrow{AC}\right|$.

2 marks

(iv) Find the minimum area of the triangle ABC.

1 mark

d. The point C now changes position while points A and B remain unchanged.

Let $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{AC} = \underset{\sim}{c}$:

(i) If $\underset{\sim}{c} = -\frac{1}{2}\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$, find parallel and perpendicular projections of $\underset{\sim}{a}$ on $\underset{\sim}{c}$.

2 marks

(ii) Hence find the area of the triangle ABC.

1 mark

Total Marks = 10

End of Paper